**Baghdad University**

**College of Sciences for Women**

**Mathematics Department**

**Third Class**

**Semester One**

# Partial Differential Equations I

## Introduction

A partial differential equation (PDE) describes a relation between an unknown function and its partial derivatives. PDEs appear frequently in all areas of physics and engineering. Moreover, in recent years we have seen a dramatic increase in the use of PDEs in areas such as biology, chemistry, computer sciences (particularly in relation to image processing and graphics) and in economics (finance). In fact, in each area where there is an interaction between a numbers of independent variables, we attempt to define functions in these variables and to model a variety of processes by constructing equations for these functions. When the value of the unknown function(s) at a certain point depends only on what happens in the vicinity of this point, we shall, in general, obtain a PDE.

In the first semester, we shall study the following three chapters:

**Chapter One**: **Definitions and Preliminaries.**

**Chapter Two**: **First Order Quasi- Linear and Linear PDEs.**

**Chapter Three**: **First order Nonlinear PDEs.**

**Chapter Four**: **Higher Order linear PDEs.**

# Chapter One
# Definitions and Preliminaries

Equations involving one or more partial derivatives of a function of two or more independent variables are called partial differential equations (PDEs). The general form of a PDE for a function $u(x_1, x_2,..., x_n)$ is :

$$F(x_1, x_2, . . . , x_n, u, u_{x1} , u_{x2}, . . . , u_{x11}, . . .) = 0 \qquad (1.1)$$

where $x_1$, $x_2$, . . . , $x_n$ are the independent variables, u is the unknown function, and $u_{xi}$ denotes the partial derivative $\frac{\partial u}{\partial x_i}$. The equation is, in general, supplemented by additional conditions such as initial conditions or boundary conditions.

Well known examples of PDEs are the following equations of mathematical physics in which the notation: $u_x = \frac{\partial u}{\partial x}$, $u_{xy} = \frac{\partial u}{\partial y \partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc., is used:

[1] One-dimensional wave equation: $u_{tt} = c^2 u_{xx}$

[2] One-dimensional heat equation: $u_t = c^2 u_{xx}$

[3] Laplace equation: $u_{xx}+u_{yy} = 0$, (2-D), or $u_{xx}+u_{yy}+u_{zz}= 0$ , (3-D)

[4] Poisson equation: $u_{xx}+u_{yy} = f(x, y )$, (2-D), or $u_{xx}+u_{yy}+u_{zz}= f(x,y,z)$ , (3-D)

**<u>Def:</u> The order** is defined to be the order of the highest derivative in the equation. If the highest derivative is of order k, then the equation is said to be of order k. Thus, for example, the equation $u_{tt} - u_{xx} = f (x, t)$ is called a second-order equation, while $u_t + u_{xxxx} = 0$ is called a fourth-order equation.

The general form of the 1st order PDE is:

$$F(x, y, z, z_x, z_y) =0 \quad , \quad \text{where} \quad z=z(x,y) \qquad (1.2)$$

$$F(x, y, z, u, u_x, u_y, z_y) =0 \quad , \quad \text{where} \quad u=u(x,y,z) \qquad (1.3)$$

The general form of the 2nd order PDE is

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0 \quad , \quad \text{where } z=z(x,y) \qquad (1.4)$$

**<u>Def:</u> The degree** of the PDE is the power of the highest ordered partial derivative in the equation. For example: wave equation: $u_{tt} = c^2 u_{xx}$ of the 2nd order and 1st degree

heat equation: $u_t = c^2 u_{xx}$ of the 2nd order and 1st degree

**<u>Def:</u>** A PDE is **linear** if the dependent variable and its derivatives are all of first degree and not multiplied together, otherwise it is nonlinear.

For example $z_x=z_{yy}$ is linear , while $z_x+zz_{yy}=\sin xy$ and $\frac{\partial z}{\partial y} = 2y(\frac{\partial z}{\partial x})^2$ are nonlinear.

The general form of the 2nd order linear PDE is

$$R(x,y)\, z_{xx}+S(x, y)\, z_{xy} + T(x, y)\, z_{yy} + P(x, y)\, z_x+ Q(x,y)\, z_y+ Z(x,y)\, z = F(x, y) \qquad (1.5)$$

Where R,S, T, P, Q, Z are functions of the independent variables x and y.

**<u>Def:</u>** <u>Nonlinear</u> PDE is said to be **quasi-linear** if it is linear in the highest ordered partial derivatives of the depended variable.

The general form of the 1st order quasi-linear PDE is

$$P(x, y, z)\, z_x + Q(x, y, z)\, z_y = R(x, y, z) \quad \text{where} \quad z=z(x, y) \qquad (1.6)$$

Where P, Q, R are functions of the dependent variable $z$ and independent variables $x$, $y$. This PDE is called **Lagrange Partial Equation**. For example $z_x+zz_{yy}=\sin xy$ is quasi linear while $\frac{\partial z}{\partial y} = 2y(\frac{\partial z}{\partial x})^2$ is nonquasi linear.

**Def:** A PDE is **homogeneous** if each term in the equation contains either the dependent variable or one of its derivatives. Otherwise, the equation is said to be **non-homogeneous**.

For example the $2^{nd}$ order linear PDE (1,5) is nonhomogeneous, if F($x$, $y$) =0 then it is homogeneous.

**Def:** The PDE is called with **constant coefficients** if all the coefficients are constants; otherwise it is with **variables coefficients**. For example $z_{xx}+2z_{xy}+z_{yy}=0$ is of constant coefficients while $x^2\, z_{xx}-y^3\, z_{yy}+z_x=0$ is of variable coefficients.

**Def:** The linear PDE is said to be of **homogeneous terms** if all the partial derivatives appearing in the equation are of the same order. For example $\frac{\partial^2 z}{\partial x^2}-x\frac{\partial^2 z}{\partial x\partial y}=4x^2$ is of homogeneous terms while $x^2\frac{\partial^3 z}{\partial x^3}+5xy\frac{\partial^3 z}{\partial x^2\partial y}+\frac{\partial^2 z}{\partial x^2}=x^2$ is of nonhomogeneous terms.

**H.W.:**

| PDE | **Order** | **degree** | **Linear** | **Quasi Linear** | **Homog-eneous** | **Constant Coefficient** | **Homo. Terms** |
|---|---|---|---|---|---|---|---|
| $z_{xx}-2z_{xy}+2z_{yy}=x+3y$ | 2 | 1 | √ | √ | x | √ | √ |
| $x^2z_{xx}-y^3z_{yy}+z_x=sinx$ | 2 | 1 | √ | √ | x | x | x |
| $xz_{yyy}=y^3z_{xx}$ | 3 | 1 | √ | √ | √ | x | x |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $z(x^2 z_y+y^2 z_x)=xy^2\cos z$ | 1 | 1 | x | √ | √ | x | - |
| $\frac{\partial z}{\partial y}=2y(\frac{\partial z}{\partial x})^2$ | 1 | 2 | x | x | √ | x | - |
| $z^2_{xx}+2z_{xy}+3z^2_{y}=xy^2z$ | 2 | 2 | x | x | √ | x | - |
| $x^2z_{xx}-z^2=0$ | 2 | 1 | x | √ | √ | x | - |
| $yu_y-2x^3y^3u_{xy}=g(x, y)$ | 2 | 1 | √ | √ | x | x | x |
| $e^{x+y}z_x+\sec x\, z_y-z=0$ | 1 | 1 | √ | √ | √ | x | √ |
| $z_xz_y-1=0$ | 1 | 1 | x | x | x | √ | - |
| $u_{xxxx}-u_{tt}=0$ | 4 | 1 | √ | √ | √ | √ | x |
| $(\frac{\partial^3 z}{\partial x^3})+2(\frac{\partial^2 z}{\partial x\partial y})+3(\frac{\partial z}{\partial y})^3+xy=0$ | 3 | 1 | x | x | x | √ | - |
| $uu_{xx}+u_y+u_{zz}=0$ | 2 | 1 | x | √ | √ | √ | - |
| $xz_{xx}+yz_{yy}=0$ | 2 | 1 | √ | √ | √ | x | √ |
| $z_x+e^{-x}z_{yy}=0$ | 2 | 1 | √ | √ | √ | x | x |
| $u_{xxy}+xu_{yy}+8u=7y$ | 3 | 1 | √ | √ | x | x | x |

**<u>Def:</u>** A **solution** of a PDE is a function $u$ such that it satisfies the equation under discussion and satisfies the given conditions as well. In other words, for $u$ to satisfy the equation, the left hand side of the PDE and the right hand side should be the same upon substituting the resulting solution. For example, $u = x^2 - y^2$, $u = e^x \cos(y)$, and $u = \ln(x^2+y^2)$, are all solutions to the two-dimension Laplace equation $u_{xx}+u_{yy} = 0$.

**Def:** A **general solution** of a PDE is a solution that contains arbitrary function equal in numbers to the order of the PDE. A **particular solution** of a PDE is a solution obtained from the general solution by particular selection of the arbitrary functions.

**Remarks**: 1- The general solution of linear ODE of order n is a family of solutions depending on n arbitrary constants. In the case of PDE, the general solution depends on arbitrary functions rather than on arbitrary constants.

2- In the case of ODE, the first step is to find the general solution and then a particular solution is determined by finding the values of arbitrary constants from the given conditions. But for PDE, selecting a particular solution satisfies the given conditions from the general may be more difficult, because the general solution of PDE contains arbitrary functions.

**Def**: The **complete solution** of PDE is a solution that contains arbitrary constants.

**Properties of arbitrary functions**:

1- If $\varphi(u)$ is an arbitrary function and A is a constant then each of $A\varphi(u)$ , $\varphi(Au)$ and $\varphi(u)+A$ is an arbitrary function.

2- If $\varphi(u)$ is an arbitrary function then $\varphi'(u)$ and $\int \varphi(u)du$ are arbitrary functions.

3- If $A_1$ and $A_2$ are arbitrary constants then $A_1$ can be written as an arbitrary function of $A_2$, that is $A_1=\varphi(A_2)$ also $A_2=\varphi(A_1)$.

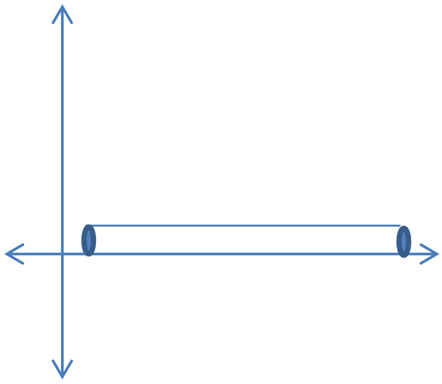
4- If $\varphi(u,v) =0$ where $\varphi$ is an arbitrary function then $u$ can be written as an arbitrary function of $v$, $u = \psi(v)$ also $v=\theta(u)$.

**Def**: **Initial value problem (IVB)** is the problem of finding an unknown function of PDE satisfying initial conditions I.C. (which describe the unknown function throughout the given region at an initial point).

**Def**: **Boundary Value Problem (BUP)** is the problem of finding an unknown function of PDE satisfying boundary conditions B.C. (which describe the unknown function at boundary points of the regions).

**Def**: **Initial-Boundary value problem (IBVP)** is the problem that consists of PDE and I.C. and B.C. The IBVPs are mathematical model of most physical phenomena.

**Example: IBVP:**



**PDE:** $U_t = U_{xx}$ , $0 < x < l$ , $t > 0$

**I.C. :** $U(x, 0) = T_o$ , $0 \le x \le l$

**B.C. :** $U(0, t) = T_1$, $t > 0$

$U(l, t) = T_2$

The heat equation in a rod of length $l$.

**Example**: Verify that $z = Ax^2 + By^2$ where A and B are arbitrary constant is a solution of the PDE: $x z_x + y z_y = 2z$ diff. with respect to $x$ and $y$ , we get:

$$\frac{\partial}{\partial x} = z_x = 2Ax$$

$$\frac{\partial}{\partial y} = z_y = 2By$$

L.H.S. $= xz_x + yz_y = x(2Ax) + y(2By) = 2Ax^2 + 2By^2 = 2(Ax^2 + By^2) = 2z$ =R.H.S

Then $z = Ax^2 + By^2$ is a solution of the given PDE

This solution is called complete solution of the PDE because it is contains arbitrary constants A & B.

**Example**: Verify that $z = \varphi(2x + y)$ where $\varphi$ is arbitrary function is a solution of the PDE $z_x - 2z_y = 0$ .

$z = \varphi(u)$, $u = 2x+y$

diff. w.r.t. x and y , we get:

$$\frac{\partial}{\partial x} = z_x = \frac{\partial \varphi}{\partial u}.\frac{\partial u}{\partial x} = \varphi'(u).2 = 2\varphi'(u)$$

$$\frac{\partial}{\partial y} = z_y = \frac{\partial \varphi}{\partial u}.\frac{\partial u}{\partial y} = \varphi'(u).1 = \varphi'(u)$$

L.H.S. = $z_x - 2z_y = 2\varphi'(u) - 2\varphi'(u) = 0$ = R.H.S.

Then $z = \varphi(2x + y)$ is a solution of PDE $z_x - 2z_y = 0$

This solution is called the general solution of the given PDE because it contains arbitrary function equal to the order of the PDE.

**Example**: Verify that $z = y^2 + y\varphi(x) + \psi(x)$ where $\varphi$ & $\psi$ are arbitrary functions is a solution of the PDE $\frac{\partial^2 z}{\partial y^2} = 2$.

diff. w.r.t. x and y , we get:

$$\frac{\partial z}{\partial y} = z_y = 2y + \varphi(x)$$

$$\frac{\partial^2 z}{\partial y^2} = z_{yy} = 2$$

Then L.H.S.=2=R.H.S.

It is a general solution of the PDE because it has arbitrary functions.

Some particular solutions:

$z = y^2 + y\sin x + x^2$ , $z = y^2 + yx^2 + 2x$ , $z = y^2 + y\ln x + x^3$

**H.W.:** Verify that: 1- $z = Ax + A^2y^2 + B$ is a solution of the PDE :
$$\frac{\partial z}{\partial y} = 2y\left(\frac{\partial z}{\partial x}\right)^2$$

2- $u = \varphi(x - ct) + \psi(x + ct)$ is the general solution of the wave equation : $u_{tt} = c^2 u_{xx}$

## 1.2.: Construction of PDE from a Solution

We can find the PDE from a given relation between the variables (which represent a solution) by:

- Elimination of arbitrary constants.
- Elimination of arbitrary functions.

**Elimination of arbitrary constants:**

Consider the dependent variable $z$ is a function of two independent variables $x$ and $y$.

**Case (1):** The number of arbitrary constants $\leq$ the number of independent variables

i.e. The given solution is defined by the relation :

$$F(x, y, z, A, B)=0 \quad \text{or} \quad F(x, y, z, A)=0 \qquad \text{……….(1.7)}$$

where A and B are arbitrary constants.

Then we can find the PDE by the following steps:

**Step (1):** Differentiating (1.7) partially with respect to $x$ and $y$, we get two equations.

**Step (2):** Eliminating the arbitrary constants from the given solution (1.7) and these two equations (from step 1) , will get the required PDE. In this case, we get one or more PDE of $1^{st}$ order.

**Case (2):** The number of arbitrary constants > the number of independent variables.

i.e. $$F(x, y, z, A, B, C,)=0 \qquad (1.8)$$

where A, B and C are arbitrary constants.

We perform step (1) as in case (1) but these resulting equations are not sufficient for the elimination of three constants. So we must obtain the partial derivatives of order two and the eliminating the arbitrary constants (A, B and C) from these equations will give the PDE.

In this case it is possible to get more than PDE which is of order higher that in the first case.

**Example 1**: Find the PDE by eliminating the arbitrary constants from each of the relations:

1- $z = Ax^2 + By^2$ where z=z(x, y)

2- $z = Ax + y$

3- $z = Ax + By + Cxy$

<u>Sol.</u> 1- $z = Ax^2 + By^2$ ….. (1)

$z_x = 2Ax$ ….. (2) $\rightarrow$ $A = \frac{z_x}{2x}$

$z_y = 2By$ …… (3) $\rightarrow$ $B = \frac{z_y}{2y}$

Then by substituting eq. 2 & 3 in 1 , we get:

$z = \frac{z_x}{2x}.x^2 + \frac{z_y}{2y}.y^2$ $\rightarrow$ $z = \frac{1}{2}xz_x + \frac{1}{2}yz_y$ is a PDE of the first order

2- $z = Ax + y$ ……(1)

$z_x = A$ ……(2) $\rightarrow$ $A = z_x$

$z_y = 1$ …….(3)

Then by substituting eq. 2 in 1, we get:

$z = x\,z_x + y$ $\rightarrow$ $x\,z_x + y = z$ is a PDE of the first order

3- $z = Ax + By + Cxy$ …… (1)

$z_x = A + C\,y$ ……(2)

$z_{xx} = 0$ is a PDE of the 2nd order

$z_{xy} = C$ ….. (3)

$z_y = B + Cx$ …… (4)

$z_{yy} = 0$ is a PDE of the 2nd order

by substituting 3 in 4 , we get: $z_y = B + x\,z_{xy}$

$\rightarrow$ $B = z_y - x\,z_{xy}$ …. (5)

by substituting 3 in 2 , we get: $z_x = A + y\, z_{xy}$

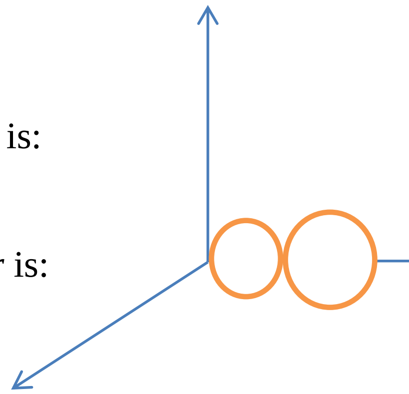
$$\rightarrow \quad A = z_x - y\, z_{xy} \quad \ldots.(6)$$

by substituting 3, 5, 6 in 1 , we get:

$z = (z_x - y\, z_{xy})\, x + (z_y - x\, z_{xy})\, y + xy\, z_{xy} \rightarrow z = x\, z_x - xy\, z_{xy} + y\, z_y - xy\, z_{xy} + xy\, z_{xy}$

$\rightarrow z = x\, z_x - xy\, z_{xy} + y\, z_y$ is a PDE of the 2nd order.

**<u>Example 2</u>:** Find the PDE of the family of spheres with centers on the x-axis.



**Sol.** The equation of the sphere with center (a,b,c) and radius $r$ is:

$(x- a)^2+(y- b)^2+(z- c)^2=r^2$

Then the equation of the sphere with center (a,0,0) and radius $r$ is:

$(x- a)^2+(y- 0)^2+(z- 0)^2=r^2$

$(x- a)^2+ y^2+ z^2=r^2 \quad \ldots\ldots\ldots.. (1)$

By derivative (1) w.r.t. $x$, we get: $2(x- a) + 2\, z\, z_x =0 \quad \ldots\ldots\ldots.(2)$

By derivative (1) w.r.t. $y$, we get: $2y + 2\, z\, z_y =0 \quad \ldots\ldots\ldots.(3)$

Then $y + z\, z_y =0$ is a PDE of the family of spheres with centers on the x-axis.

**<u>H.W.1</u>** : Find the PDE by eliminating the arbitrary constants from each of the relations:

1) $z = A(x + y)$  2) $z = Ax + A^2y^2 + B$  3) $Ax + By + Cz =1$

**<u>H.W.2</u>**: Find the PDE whose solution is all spheres with centers on the x-axis and of radius1.

## Elimination of arbitrary functions:

Consider the dependent variable $z$ is a function of two independent variables $x$ and $y$.

**Case (1):** The solution contains $n$ arbitrary functions of the form $\varphi(u)$ where $u = u(x, y)$.

**Step (1):** Differentiate the solution partially w.r.t. $x$ & $y$ (we must find the partial derivatives of order $n$).

**Step (2):** Eliminating the $n$ arbitrary functions and their derivatives from the resulting equations will give PDE of order $n$.

**Example:** Find the PDE whose general solution is given by the relation:

1- $z=\varphi(x)+\psi(y)$  2- $z=x^2\psi(x-y)$

**Sol.**

1- $z=\varphi(x)+\psi(y)$

$z_x=\varphi'(x)$ & $z_y=\psi'(y)$

$z_{xx}=\varphi''(x)$ & $z_{yy}=\psi''(y)$ & $z_{xy}=0$ is a PDE of order 2.

2- $z=x^2\psi(x-y)$ .....(1)

$z_x=2x\,\psi(x-y)+x^2\psi'(x-y)$ .....(2)

$z_y=-x^2\psi'(x-y) \rightarrow \psi'(x-y)=-\dfrac{z_y}{x^2}$ .....(3)

by substituting (3) in (2), we get:

$z_x=2x\,\psi(x-y)-x^2\dfrac{z_y}{x^2} \rightarrow \psi(x-y)=\dfrac{z_x+z_y}{2x}$ .....(4)

by substituting (4) in (1), we get:

$\dfrac{2z}{x}=z_x+z_y$ is a PDE of the 1st order.

**Case (2):** Where the solution is given by the relation $\varphi(u, v)=0$ where $u=u(x, y, z)$ and $v=v(x, y, z)$.

Then differentiating the relation partially w.r.t. x & y respectively, we get:

$$\frac{\partial\varphi}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}z_x\right]+\frac{\partial\varphi}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}z_x\right]=0 \quad .....(1)$$

$$\frac{\partial\varphi}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}z_y\right]+\frac{\partial\varphi}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}z_y\right]=0 \quad .....(2)$$

eliminating $\frac{\partial\varphi}{\partial u}$ & $\frac{\partial\varphi}{\partial v}$ from (1) & (2), we have:

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}z_x & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}z_x \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}z_y & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}z_y \end{vmatrix}=0$$

$$\rightarrow [\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}z_x][\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}z_y]-[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}z_x][\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}z_y]=0$$

$\rightarrow$ **$(u_yv_z-u_zv_y)z_x+(u_zv_x-u_xv_z)z_y=(u_xv_y-u_yv_x)$**

This PDE is of the form $Pz_x+Qz_y=R$ (Lagrange PDE).

**<u>Example</u>**: Find the PDE whose solution is given by the relation:

1- $\varphi(\frac{z}{x^2}, x-y)=0$ 2- $\varphi(x^2-z^2, x^3-y^3)=0$

**Sol.** 1- $\varphi(\frac{z}{x^2}, x-y)=0 \rightarrow u(x,y,z)=\frac{z}{x^2} \quad ,v(x,y,z)=x-y$

$\rightarrow u_x=\frac{-2z}{x^2} \quad ,u_y=\frac{z_y}{x^2}, \quad u_z=\frac{1}{x^2}$

& $v_x=1, \quad v_y=-1, \quad v_z=0$

$(u_yv_z-u_zv_y)z_x+(u_zv_x-u_xv_z)z_y=(u_xv_y-u_yv_x)$

$\rightarrow (\frac{z_y}{x^2}.0-\frac{1}{x^2}.(-1))\,z_x+(\frac{1}{x^2}.1-\frac{-2z}{x^2}.0)\,z_y=(\frac{-2z}{x^2}(-1)-\frac{z_y}{x^2}.1)$

$\rightarrow \frac{1}{x^2}z_x+\frac{1}{x^2}z_y=\frac{2z}{x^2}-\frac{z_y}{x^2} \quad \rightarrow \quad z_x+2z_y=2z$

2- $\varphi(x^2-z^2, x^3-y^3)=0 \rightarrow u(x,y,z)=x^2-z^2 \quad ,v(x,y,z)=x^3-y^3$

$u_x=2x, \quad u_y=0, \quad u_z=-2z$

$v_x=3x^2, \quad v_y=-3y^2, \quad v_z=0$

$(u_yv_z-u_zv_y)\,z_x+(u_zv_x-u_xv_z)\,z_y=(u_xv_y-u_yv_x)$

$\rightarrow$ $(0-(-2z)(-3y))\,z_x+((-2z)(3x^2)-0)\,z_y=((2x)(-3y)-0)$

$\rightarrow -6zy^2\,z_x-6x^2z\,z_y=-6xy^2$ $\rightarrow 6zy^2\,z_x+6x^2z\,z_y=6xy^2$

**H.W**: Find the PDE whose solution is given by the relation:

1- $\frac{z}{x}=\varphi(\frac{y}{x})$ 2- $\ln z=y+\varphi(x-y)$ 3- $u=\varphi(x-ct)+\psi(x+ct)$

4- $\varphi(\frac{x}{y},\frac{z}{x^3})=0$ 5- $\varphi(x^2+y^2,z)=0$ 6- $\varphi(x-y,\ln z-y)=0$ 7- $z=Ax^2+\varphi(y)$

# Chapter Two

## First Order Quasilinear and Linear PDE

In this chapter we shall solve the first order quasilinear PDE by using the method of characteristics due to Lagrange.

The $1^{st}$ order linear PDE is of the form:

$P(x, y)\, z_x + Q(x, y)\, z_y + Z(x, y)\, z = f(x, y)$ where $z=z(x, y)$ (2.1)

The $1^{st}$ order quasilinear PDE is of the form:

$P(x, y, z)\, z_x + Q(x, y, z)\, z_y = R(x, y, z)$ where $z=z(x, y)$ (2.2)

and it is called Lagrange Partial Equation.

## ***2.1: Solving Quasilinear PDE (Lagrange PDE)***

The general solution of PDE in (2.2) is $\varphi(u_1, u_2)=0$ where $\varphi$ is an arbitrary function and $u_1(x, y, z)= A_1$ , $u_2(x, y, z)= A_2$ are two independent solution of the ODE:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

these first order ODE is called Auxiliary equations ore Lagrange system.

* **Method of Solution the Lagrange System**:

Step(1): Write down the auxiliary equations (Lagrange system):

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Step (2): Find two independent solutions of Lagrange system say:

$u_1(x, y, z)= A_1$ ,

$u_2(x, y, z)= A_2$

Step (3): Set $\varphi(u_1, u_2)=0$ where $\varphi$ arbitrary function and this represent the general solution of Lagrange PDE.

Also it has the form : $u_1= \psi(u_2)$ or $u_2 = \psi(u_1)$

Set $u_1= Au_2+B$ or $u_1= Au_2+B$ where A & B an arbitrary constants and this represent the complete solution of Lagrange PE.

**Remark:** Lagrange method may be extended to solve 1st order quasilinear PDE containing more than two independent variables.

*__Geometrical interpretation of the general solution__:

The general solution of ODE represent integral curves in *xy*-plane $R^2$, but the general solution of PDE represent integral surface in $(x, y, z)$-space $R^3$.

**__Example:__** Find the general solution of the Lagrange DEs :

1- $xz_x-yz_y-3z=0$ 2- $3z_x+4z_y=2$ 3- $z_x+z_y=\frac{2z}{x}$

**Sol.:** 1) $xz_x-yz_y-3z=0 \rightarrow xz_x-yz_y=3z \rightarrow P=x,\ Q=-y\ \&\ R=3z$ →The Lagrange system :

$$\frac{dx}{x}=\frac{dy}{-y}=\frac{dz}{3z}$$ , to find two solution of Lagrange system:

$$\frac{dx}{x}=\frac{dy}{-y} \rightarrow \ln x = -\ln y + \ln c \rightarrow \ln x + \ln y = \ln c \rightarrow xy = A_1$$

$\rightarrow u_1 = xy = A_1$

$$\frac{dx}{x}=\frac{dz}{3z} \rightarrow \frac{dz}{z}-3\frac{dx}{x}=0 \rightarrow \ln z - 3\ln x = \ln c \rightarrow \frac{z}{x^3}=A_2 \rightarrow u_2=\frac{z}{x^3}=A_2$$

Then the general solution is : $\varphi\left(xy, \frac{z}{x^3}\right)=0$ where $\varphi$ is an arbitrary function.

The complete solution is : $u_1=Au_2+B \rightarrow xy = A\frac{z}{x^3}+B$ or $u_2=Au_1+B \rightarrow \frac{z}{x^3}=Axy+B$ where A & B are arbitrary constants.

2) $3z_x+4z_y=2 \rightarrow P=3,\ Q=4\ \&\ R=2$ →The Lagrange system:

$$\frac{dx}{3}=\frac{dy}{4}=\frac{dz}{2}$$ , to find two solution of Lagrange system:

$$\frac{dx}{3}=\frac{dy}{4} \rightarrow \frac{1}{3}x-\frac{1}{4}y=A_1 \rightarrow u_1=\frac{1}{3}x-\frac{1}{4}y=A_1$$

$$\frac{dx}{3}=\frac{dz}{2} \quad \longrightarrow \quad \frac{1}{3}x-\frac{1}{2}z=A_2 \quad \longrightarrow \quad u_2=\frac{1}{3}x-\frac{1}{2}z=A_2$$

Then the general solution is : $\varphi$ $(\frac{1}{3}x-\frac{1}{4}y, \frac{1}{3}x-\frac{1}{2}z)=0$ where $\varphi$ is an arbitrary function.

The complete solution is : $u_1=Au_2+B \rightarrow \frac{1}{3}x-\frac{1}{4}y = A(\frac{1}{3}x-\frac{1}{2}z) + B$

or $u_2=Au_1+B \rightarrow \frac{1}{3}x-\frac{1}{2}z=A(\frac{1}{3}x-\frac{1}{4}y)+B$ where A & B are arbitrary constants.

3) $z_x+z_y=\frac{2z}{x} \rightarrow P = 1, Q = 1$ & $R = \frac{2z}{x}$ →The Lagrange system:

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{\frac{2z}{x}}$$ , to find two solution of Lagrange system:

$$\frac{dx}{1}=\frac{dy}{1} \rightarrow x - y = A_1 \rightarrow u_1= x - y = A_1$$

$$\frac{dx}{1}=\frac{dz}{\frac{2z}{x}} \rightarrow \frac{dx}{1}=\frac{xdz}{2z} \rightarrow \frac{2dx}{x}=\frac{dz}{z} \rightarrow \frac{dz}{z}-\frac{2dx}{x}=0 \rightarrow \ln z - 2\ln x = c \rightarrow u_2=\frac{z}{x^2}=A_2$$

Then the general solution is : $\varphi$ ( $x$-$y$, $\frac{z}{x^2}$) =0 where $\varphi$ is an arbitrary function.

The complete solution is : $u_1=Au_2+B \rightarrow x-y = A(\frac{z}{x^2}) + B$

or $u_2=Au_1+B \rightarrow \frac{z}{x^2} =A( x-y )+B$ where A & B are arbitrary constants.

**<u>Remark</u>**: If any coefficient $P$, $Q$ or $R$ in the equation equal to zero then we put $dx$, $dy$ or $dz$ respectively equlal to zero. For example : if Q =0 then the Lagrange system is:

$\frac{dx}{P}=\frac{dy}{0}=\frac{dz}{R} \rightarrow dy = 0 \rightarrow y= c \rightarrow u_1=y=A_1$ and we get $u_2=A_2$ from $\frac{dx}{P}=\frac{dz}{R}$.

**H.W.:** Find the general solution of the Lagrange DEs :

1- $xz_x+2z=0$ 2- $(2x+1)z_y-z_x=0$ 3- $xz_x-yz_y = 0$

**ملاحظة** : بعض خواص التناسب التي نحتاجها في ايجاد حلول المعادلات المساعدة لمعادلة لاكرانج الجزئية :

١- عند ضرب النسب الثلاث ( بسط ومقام) بدوال مثل $n, m, l$ على التوالي وهي دوال بالمتغيرات $x, y, z$ فان:

$$\frac{ndx}{nP}=\frac{mdy}{mQ}=\frac{ldz}{lR}$$

وهذه النسب الثلاث تساوي النسب الاصلية بالاضافة الى انها تساوي مجموعها، اي ان:

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ndx+mdy+ldz}{nP+mQ+lR}$$

٢- باختيار معين للدوال $n, m, l$ نستطيع ان نجد معادلة تفاضلية اعتيادية ( بمتغيرين) وذلك باخذ النسبة الجديدة ( الرابعة) مع احدى النسب الثلاث .

٣- باختيار معين للدوال $n, m, l$ نستطيع ان نجعل مقام النسبة الجديدة يساوي صفر فيصبح البسط لهذه النسبة يساوي صفر اي نجعل : $nP+mQ+lR = 0$ فيكون $ndx+mdy+ldz=0$ وهي تمثل ODE يمكن حلها بالتكامل المباشر.

**Example:** Find the general solution of the PDEs:

1) $z(xz_x-yz_y) = y^2- x^2$ 2) $x(y-z)z_x+y(z-x)z_y=z(x-y)$

**Sol.** 1) $z(xz_x-yz_y) = y^2- x^2 \rightarrow xz\,z_x-yz\,z_y= y^2- x^2 \rightarrow$ P = $xz$ , Q= $-yz$ , R = $y^2- x^2$

Then Lagrange system: $\frac{dx}{xz}=\frac{dy}{-yz}=\frac{dz}{y^2-x^2}$

من خواص النسب يمكننا حذف الكميات المتساوية من البسط او المقام وبذلك يمكننا حذف z من مقامي النسبتين الاولى والثانية:

$\frac{dx}{xz}=\frac{dy}{-yz} \rightarrow \frac{dx}{x}=\frac{dy}{-y} \rightarrow \ln x = -\ln y+\ln A_1 \rightarrow \ln x+\ln y=\ln A_1 \rightarrow \ln xy =\ln A_1 \rightarrow xy = A_1$

Then $u_1= xy = A_1$

To find $u_2$:

**Method (1)**:

باستخدام الملاحظة اعلاه وبضرب النسبة الاولى ب $x$ والثانية ب $y$ والثالثة ب $z$ وجمعهم ينتج المقام:

$x(xz)-y(yz)+z(y^2-x^2)= x^2z-y^2z+y^2z-x^2z =0$

وايضا وباستخدام خواص النسب فان:

$x\,dx + y\,dy + z\,dz =0 \;\rightarrow\; \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c \;\rightarrow\; x^2+y^2+z^2=c \;\rightarrow\; u_2 = x^2+y^2+z^2 = A_2$

Then the general solution: $\varphi(\,xy,\, x^2+y^2+z^2)=0$

**<u>Method(2):</u>**

بضرب النسبة الاولى ب $-x$ والثانية ب $-y$ وجمعهما:

$$\frac{-xdx}{-x^2z}=\frac{-ydy}{y^2z}=\frac{dz}{y^2-x^2}\rightarrow\frac{-xdx-ydy}{-x^2z+y^2z}=\frac{dz}{y^2-x^2}\rightarrow\frac{-xdx-ydy}{z(y^2-x^2)}=\frac{dz}{y^2-x^2}\rightarrow\frac{-xdx-ydy}{z}=dz$$

$\rightarrow z\,dz= -xdx - ydy \;\rightarrow\; xdx + ydy + zdz=0 \rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c \;\rightarrow\; x^2+y^2+z^2=c$

$\rightarrow\; u_2 = x^2+y^2+z^2 = A_2$

Then the general solution: $\varphi(\,xy,\, x^2+y^2+z^2)=0$ where $\varphi$ is an arbitrary function.

**Sol**. 2) $x(y-z)z_x+y(z-x)z_y=z(x-y) \;\rightarrow$ P= $x(y-z)$ , Q= $y(z-x)$ & R= $z(x-y)$

Then Lagrange system: $\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$

بما ان نسبة مجموع بسطي المعادلة الاولى الى مجموع مقاميها يساوي كلا من النسب اعلاه فان

$$\frac{dx+dy}{x(y-z)+y(z-x)}=\frac{dz}{z(x-y)}\rightarrow\frac{dx+dy}{xy-xz+yz-xy}=\frac{dz}{zx-zy}\rightarrow\frac{dx+dy}{-xz+yz}=\frac{dz}{zx-zy}$$

$\rightarrow dx+dy=-dz \;\rightarrow\; dx + dy + dz=0 \rightarrow\; x+y+z=c \rightarrow\; u_1 = x+y+z = A_1$

لايجاد الحل الثاني نضرب اطراف المعادلتين التابعتين ب xyz فان:

$$\frac{xyzdx}{x(y-z)}=\frac{xyzdy}{y(z-x)}=\frac{xyzdz}{z(x-y)} \longrightarrow \frac{yzdx}{(y-z)}=\frac{xzdy}{(z-x)}=\frac{xydz}{(x-y)}$$

وبما انه نسبة مجموع بسطي المعادلة الاولى الى مجموع مقاميها يساوي كلا من احدى النسب فان:

$$\frac{yzdx+xzdz}{(y-z)+(z-x)}=\frac{xydz}{(x-y)} \longrightarrow \frac{yzdx+xzdz}{y-x}=\frac{xydz}{x-y} \longrightarrow yzdx+xzdz=-xydz \rightarrow$$

$yzdx+xzdz+xydz=0$

$d(xyz)=0 \rightarrow xyz=A_2$

Then the general solution: $\varphi(x+y+z, xyz)=0$ where $\varphi$ is an arbitrary function.

***H.W.*** Find the general solution of the PDEs:

1) $x^2z_x+y^2z_y=z^2$  2) $(y^3x-2x^4)z_x+(2wy^4-x^3y)z_y=z(x^3-y^3)$

## *2.2: Cauchy Problem*

Cauchy problem is the problem of finding the particular solution of 1st order PDE satisfying given condition (Cauchy Data).
**Or** Cauchy problem is the problem of finding the integral surface of 1st order PDE passing through given curve in (parametric form).

***Method of Solving Cauchy Problem**

To find the particular integral surface $z=z(x, y)$ of the PDE:

$P\,z_x+Q\,z_y=R$
which passing through the curve:

$[\ x=x(t)\ ,\ y=y(t)\ ,\ z=z(t)\ ]$ …..(2.3)

**Step(1)** Solving Lagrange system $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ to obtain the characteristic curves:

$u_1(x, y, z)=A_1$ & $u_2(x, y, z)=A_2$ ………..(2.4)

**Step(2)** Substituting $x, y, z$ from (2.3) into (2.4), we get:

$u_1(t) = A_1$ & $u_2(t) = A_2$ ...........(2.5)

**Step(3)** Eliminating t from (2.5) to get a relation between $A_1$ & $A_2$.

Step (4) Setting the value of $A_1$ & $A_2$ from (2.4) in this relation , we get the solution of Cauchy problem.

**<u>Note</u>**: If the given curve is in terms of $x$, $y$, $z$ then we must write it in parametric form (in terms of $t$).

**<u>Example(1):</u>** Find the solution of the following Cauchy problem:

$$z_x+z_y=z \quad \text{with} \quad [x=t,\ y=0,\ z=\cos t] \ \ldots..(1)$$

Or find the particular integral surface of the PDE which satisfying Cauchy data.

**<u>Sol.</u>** P= 1, Q = 1 & R=z, then the Lagrange system:

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{z}$$

$dx = dy \rightarrow x=y+c \rightarrow x\text{-}y = A_1$ .....(2)

$dy = \frac{dz}{z} \rightarrow y - \ln z = A_2$ ......(3)

substituting (1) in (2) & (3), we get:

$t - 0 = A_1 \rightarrow t = A_1$

$0 - \ln(\cos t) = A_2 \rightarrow -\ln(\cos A_1) = A_2 \rightarrow A_2 = -\ln(\cos A_1)$

Setting the values of $A_1$ & $A_2$ from (2) & (3), we get:
$y - \ln z = -\ln(\cos(x\text{-}y)) \rightarrow \ln z ,= y + \ln(\cos(x\text{-}y)) \xrightarrow{e} z= e^y \cos(x-y)$ integral surface passing through the curve.

**<u>Example(2):</u>** Find the general solution of the PDE $(2x+1)z_y\text{-}z_x=0$ and then find the particular solution that satisfying the condition $z(1, y)=y-4$ .

**Sol.** P = -1, Q =2$x$ + 1 & R =0, then Lagrange system: $\frac{dx}{-1}=\frac{dy}{2x+1}=\frac{dz}{0}$

$dz = 0 \rightarrow z= A_1$ , ......(1)

$\frac{dx}{-1}=\frac{dy}{2x+1} \rightarrow (2x+1)\,dx = -dy \rightarrow x^2 + x + y = A_2$ ……(2)

Then the general solution: $\varphi(z, x^2 + x + y) = 0$

The given curve in parametric form: [ $x=1, y=t, z=t-4$] ……..(3)

By substituting (1) & (2) in (3), we get:

$t - 4 = A_1$

$1+1+t = A_2 \rightarrow 2+t = A_2$

$A_2 - A_1 = 6 \rightarrow x^2 + x + y - z = 6$ (بطرح ١ من ٢)

**H.W.** Find the particular integral surface of the following PDE which satisfying Cauchy data:

(1) $4yz\, z_x + z_y + 2y = 0$ , $x+z=2, y^2+z^2=1$

(2) $y\, z_x = x$, $x=0, z=y^3$

(3) $2z_x - xz_y = xz$ , $x=t, y=t^2, z=t^3$

(4) $z_x - z_y = 1, z(x,0) = x^2$

**Sol.** 1) $4yz\, z_x + z_y + 2y = 0 \rightarrow 4yz\, z_x + z_y = -2y \rightarrow P = 4yz$ , $Q=1$ & $R=-2y$

Then Lagrange System: $\frac{dx}{4yz}=\frac{dy}{1}=\frac{dz}{-2y}$

$\frac{dy}{1}=\frac{dz}{-2y} \rightarrow -2y\,dy = dz \rightarrow dz + 2y\,dy = 0 \rightarrow z + y^2 = A_1$ …..(1)

$\frac{dx}{4yz}=\frac{dz}{-2y} \rightarrow \frac{dx}{4yz}=\frac{2z\,dz}{-4zy} \rightarrow \frac{dx+2z\,dz}{4yz-4zy}=0 \rightarrow dx + 2z\,dz = 0 \rightarrow x + z^2 = A_2$ ….(2)

The general solution is: $\varphi(z+y^2, x+z^2) = 0$

To find the particular solution

$x+z=2 \rightarrow x = 2-z$ , $y^2+z^2=1 \rightarrow y=\sqrt{1-z^2}$

let $z=t$ , $x = 2-t$ , $y=\sqrt{1-t^2}$ …..(3)

substitute (3) into (1) & (2), we get:

$t + 1 - t^2 = A_1$ & $2 - t + t^2 = A_2$

by adding , we get: $A_1 + A_2 = 3$

$\to z + y^2 + x + z^2 = 3 \to (z^2 + x) + (y^2 + z) - 3 = 0$ is a particular integral surface to the PDE.

## 2.3. Surface Orthogonal to Given Family of Surfaces

To find equation of surface orthogonal to given family of surfaces $f(x, y, z) = c$ [i.e. the surface which intersect each of the given surfaces at right angles (orthogonally)]:

(1) Differentiate $f(x, y, z)$ partially w.r.t. $x$, $y$ & $z$ to get:

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$$

(2) Solve the PDE ( Lagrange PDE):

$$\frac{\partial f}{\partial x} z_x + \frac{\partial f}{\partial y} z_y = \frac{\partial f}{\partial z}$$

{which determine the surface orthogonal to the given family of surfaces}.

To obtain particular solution which represent the orthogonal surface.

**Example:** Find the equation of the surface passing through $y=x$, $z=1$ and orthogonal to the family of surfaces $x^2+y^2+z^2=cx$.

**Sol.** $x^2+y^2+z^2=cx \to \dfrac{x^2+y^2+z^2}{x} = c \;\to\; f(x, y, z) = \dfrac{x^2+y^2+z^2}{x}$

$$\frac{\partial f}{\partial x} = \frac{x(2x) - (x^2 + y^2 + z^2)}{x^2} = \frac{x^2 - y^2 - z^2}{x^2}, \qquad \frac{\partial f}{\partial y} = \frac{2y}{x}, \qquad \frac{\partial f}{\partial y} = \frac{2z}{x}$$

The PDE: $\dfrac{x^2 - y^2 - z^2}{x^2} z_x + \dfrac{2y}{x} z_y = \dfrac{2z}{x}$

The Lagrange system : $\dfrac{dx}{\dfrac{x^2-y^2-z^2}{x^2}}=\dfrac{dy}{\dfrac{2y}{x}}=\dfrac{dz}{\dfrac{2z}{x}} \rightarrow \dfrac{x^2dx}{x^2-y^2-z^2}=\dfrac{xdy}{2y}=\dfrac{xdz}{2z}$

$\dfrac{xdy}{2y}=\dfrac{xdz}{2z} \rightarrow \dfrac{dy}{y}=\dfrac{dz}{z} \rightarrow \ln y = \ln z + \ln c \rightarrow \ln y - \ln z = \ln c \rightarrow \ln\dfrac{y}{z}=\ln c$

$\rightarrow \dfrac{y}{z}=A_1$ ….(1)

بضرب النسبة الاولى ب ١ والثانية ب y والثالثة ب z وبجمع النسب واخذ النسبة الجديدة مع النسبة الثالثة:

$$\frac{x^2dx}{x^2-y^2-z^2}=\frac{xydy}{2y^2}=\frac{xzdz}{2z^2}$$

$\rightarrow \dfrac{x^2dx+xydy+xzdz}{x^2-y^2-z^2+2y^2+2z^2}=\dfrac{xdz}{2z} \rightarrow \dfrac{x(xdx+ydy+zdz)}{x^2+y^2+z^2}=\dfrac{xdz}{2z} \rightarrow \dfrac{(xdx+ydy+zdz)}{x^2+y^2+z^2}=\dfrac{dz}{2z}$

$\rightarrow \dfrac{2(xdx+ydy+zdz)}{x^2+y^2+z^2}=\dfrac{dz}{z} \rightarrow \ln(x^2+y^2+z^2) = \ln z + \ln c \rightarrow \ln(x^2+y^2+z^2) - \ln z = \ln c$

$\rightarrow \ln\dfrac{(x^2+y^2+z^2)}{z}=\ln c \rightarrow \dfrac{(x^2+y^2+z^2)}{z}=A_2$ …..(2)

Then the general solution : $\psi(\dfrac{y}{z})=\dfrac{x^2+y^2+z^2}{z}$

$y=x,\ z=1 \rightarrow [\,x=t,\ y=t,\ z=1\,]$ ….(3)

substitute (3) into (1) & (2), we get:

$A_1=\dfrac{t}{1}=t$ , $A_2=\dfrac{(t^2+t^2+1^2)}{1}=2t^2+1 \rightarrow A_2=2A_1^2+1 \rightarrow 2A_1^2-A_2+1=0$

$\rightarrow 2(\dfrac{y}{z})^2-\dfrac{x^2+y^2+z^2}{z}+1=0$ is the equation of the surface orthogonal to given surface.

**<u>H.W.</u>** 1) Find the equation of the surfaces orthogonal to the family of surfaces: $z^2=\underline{cxy}$.

2) Find the equation of the surface passing through $x^2+y^2=1$, $z=1$ and orthogonal to the family of surfaces $z(x+y)=c(z+1)$.